M303/B

M3031706F1PV1

Module Examination 2017

Further Pure Mathematics

Friday 2 June 2017 **2.30 pm – 5.30 pm**

Time allowed: 3 hours

The exam is in two parts. Attempt *all* questions in Part 1 (each question is worth 11 marks) and *two* out of the three questions in Part 2 (each question is worth 17 marks). If you submit attempts to all three questions in Part 2, your best two scores will count towards your result.

It should take about two hours to do the Part 1 questions and an hour to do two Part 2 questions.

Start each question on a new page.

When sketching sets, take care to indicate which points are included: use dashed lines to indicate boundaries that are not included and solid lines to indicate boundaries that are. Use small open circles ∘ to indicate any 'corner' points on a boundary that are not included and small closed circles • to indicate any 'corner' points that are included.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.

The use of calculators is *not* permitted in this examination.

N.B. Q3(c) contained a typo. We believe the correct expression for $f(x)$ should have been

$$f(x) = x^4 + x^3 + 16x^2 - 49x + 36$$

not

$$f(x) = x^4 + x^3 + 16x^2 - 49x + 34.$$

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PART 1

Attempt all questions. Each question is worth 11 marks. All of the Part 1 questions count towards your result.

Question 1

(a) (i) Use the Euclidean algorithm to find $\text{hcf}(143, 312)$. [3]

(ii) Hence find integers s and t such that

$$143s + 312t = \text{hcf}(143, 312).$$ [4]

(b) Prove by induction that

$$\sum_{i=1}^{n}(3i - 2) = \frac{n(3n - 1)}{2}$$

for all positive integers n. [4]

Question 2

(a) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$.

(i) Write down the order of G. [1]

(ii) Write down the order of $(0, 3)$ in G. [1]

(iii) Write down an element of order 6 in G. [1]

(iv) Is G cyclic? Justify your answer. [2]

(b) Consider the dihedral group D_{12} given by

$$D_{12} = \langle r, s | r^{12} = s^2 = e, sr = r^{11}s \rangle.$$

(i) Write the element r^6sr^6 in the form r^is^j for some $i \in \{0, \ldots, 11\}$ and some $j \in \{0, 1\}$. [2]

(ii) Find the order of the element r^6s. [2]

(iii) Write down a subgroup of D_{12} of order 12. [1]

(iv) Is this subgroup normal? Justify your answer. [1]

Question 3

(a) Given $75 = 3 \times 5^2$, find

(i) $\tau(75)$,

(ii) $\sigma(75)$,

(iii) $\phi(75)$. [3]

(b) Calculate the following Legendre symbols (note that 179 and 199 are prime)

(i) $(2/199)$,

(ii) $(179/199)$. [4]

(c) Let $f(x) = x^4 + x^3 + 16x^2 - 49x + 34$ be a polynomial in $\mathbb{Q}[x]$.

(i) Show that

$$f(x+1) = x^4 + 5x^3 + 25x^2 - 10x + 5.$$ [2]

(ii) Hence, or otherwise, prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$. [2]

Question 4

Let
$A = \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4\}$,
$B = \{(a, b) \in \mathbb{R}^2 : |b| < 1\}$ and
$C = A \cap B$.

(a) Sketch on separate diagrams or describe in words each of A, B and C taking care to indicate which points lie in the sets. [3]

(b) Write down the closure, interior and boundary of C for the Euclidean metric. [3]

(c) Show that C is not connected for the Euclidean metric. [3]

(d) Determine the d_0-closure of $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ in $\mathbb{R}$. [2]

Question 5

Let $\alpha = \sqrt{7} - \sqrt{11}$; you may assume that
$[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{11}) : \mathbb{Q}] = 2$, $\sqrt{7} \notin \mathbb{Q}(\sqrt{11})$ and $\sqrt{11} \notin \mathbb{Q}(\sqrt{7})$.

(a) Express α^3 in the form $a\sqrt{7} + b\sqrt{11}$ for suitable $a, b \in \mathbb{Q}$. [2]

(b) With the help of (a) or otherwise, show that $\sqrt{7}, \sqrt{11} \in \mathbb{Q}(\alpha)$. [3]

(c) Hence show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{7}, \sqrt{11})$. [1]

(d) Determine the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. [2]

(e) By considering α^2 find the minimal polynomial for α over $\mathbb{Q}$. [3]

Question 6

Let $A = [0, 2)$ and $B = \{0\} \cup \{2\}$.

(a) For each of the sets A and B, state whether or not they are d-connected, d-compact and d-complete when:

(i) d is the Euclidean metric for $\mathbb{R}$; [3]

(ii) d is the discrete metric for $\mathbb{R}$. [3]

(b) Give brief justifications of your answers for the sets A and B in the case when d is the discrete metric. [5]

Part 2

Attempt two questions. Each question is worth 17 marks. (If you submit attempt all three questions, your best two scores will count towards your result.)

Question 7

Let G be a group of order $325 = 13 \times 5^2$.

(a) (i) Write down the order of the Sylow 5-subgroups of G. [*1*]

(ii) Show that G has a unique Sylow 13-subgroup and a unique Sylow 5-subgroup. [*3*]

(iii) Deduce that G is abelian. [*5*]

(iv) Write down the non-isomorphic possibilities for G. [*2*]

(b) Let G be a non-abelian group of order p^3 where p is prime.

(i) Prove that the centre is isomorphic to $\mathbb{Z}_p$. [*4*]

(ii) Deduce that G has a normal subgroup of order p^2 (Hint: use the correspondence theorem.) [*2*]

Question 8

Let $R = \mathbb{Z}[\sqrt{-7}] = \{a + b\sqrt{-7} : a, b \in \mathbb{Z}\}$, and let

$$N(a + b\sqrt{-7}) = a^2 + 7b^2$$

be a norm on R, which you may assume is multiplicative.

(a) Show that the only elements $r \in R$ with $N(r) = 1$ are $r = \pm 1$, and that there are no elements with $N(r) = 2$. [*3*]

(b) Show that all of 2, $1 + \sqrt{-7}$ and $1 - \sqrt{-7}$ are irreducible in R. [*5*]

(c) Hence, or otherwise, show that R is not a UFD. [*1*]

Let $I = \langle 2, 1 + \sqrt{-7} \rangle$ be the ideal in $\mathbb{Z}[\sqrt{-7}]$ generated by the elements 2 and $1 + \sqrt{-7}$.

(d) Show that if $\alpha + \beta\sqrt{-7} \in I$ for $\alpha, \beta \in \mathbb{Z}$, then $\alpha - \beta$ is even. [*2*]

(e) By considering the cases where $\alpha, \beta \in \mathbb{Z}$ are both even or both odd separately, prove that if $\alpha - \beta$ is even, then $\alpha + \beta\sqrt{-7} \in I$.

Conclude that $I = \{\alpha + \beta\sqrt{-7};\ \alpha, \beta \in \mathbb{Z},\ \alpha - \beta \text{ is even}\}$. [*3*]

(f) With the help of (e) or otherwise, show that for every $x \in \mathbb{Z}[\sqrt{-7}]$ such that $x \notin I$ we have $1 + x \in I$. [*1*]

(g) Using (f) or otherwise, show that I is a maximal ideal in $\mathbb{Z}[\sqrt{-7}]$. [*2*]

Question 9

Define a distance function $d\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ as follows:

$$d(a,b) = \begin{cases} 0, & \text{if } a-b=0, \\ 1, & \text{if } a-b \text{ is even and non-zero}, \\ 2, & \text{if } a-b \text{ is odd}. \end{cases}$$

(a) Prove that d is a metric on $\mathbb{Z}$ by showing that

(i) d satisfies (M1). [2]

(ii) d satisfies (M2). [3]

(iii) d satisfies (M3). [5]

(b) Show that if a sequence (a_i) in $\mathbb{Z}$ is eventually constant, then it is d-Cauchy. [2]

(c) Show that if a sequence (a_i) in $\mathbb{Z}$ is d-Cauchy, then it is eventually constant. [3]

(d) Briefly explain whether $(\mathbb{Z}, d)$ is a complete metric space. [2]

[END OF QUESTION PAPER]